## On the Translational Part of the Lagrangian in the Poincaré Gauge Theory of Gravitation

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Z. Naturforsch. **33a**, 1224–1225 (1978); received July 14, 1978

Using differential forms we postulate a guiding principle yielding the quadratic Lagrangian of von der Heyde.

Recently Hehl, Ne'eman, Nitsch and von der Heyde [1] discussed several physical aspects of the Poincaré gauge theory of gravitation with a quadratic Lagrangian first proposed by von der Heyde [2]. In terms of differential forms (we adopt the notation and conventions of [3]) this Lagrangian reads as follows:

$$L = L_{\text{transl}} + L_{\text{rot}}\,, \tag{1}$$

$$L_{\text{transl}} = -\frac{1}{2l^2}\,(\Theta^a \wedge e^b) \wedge * (\Theta_b \wedge e_a)\,, \tag{2}$$

$$L_{\text{rot}} = \frac{1}{2k}\,\Omega^a{}_b \wedge * \Omega^b{}_a = -\frac{1}{2k}\,\Omega^{ab} \wedge * \Omega_{ab}\,. \tag{3}$$

Here $l$ is the Planck length and $k$ a dimensionless "strong" coupling constant. By $\Theta^a$ and $\Omega^a{}_b$ we denoted the field strengths corresponding to the translational and rotational gauge potentials $e^a$ and $\omega^a{}_b$. These may be interpreted in geometrical terms as an orthonormal tetrad and a connection in a Riemann-Cartan space $U_4$. The field strengths are thus identified with torsion and curvature, respectively:

$$\Theta^a = d\,e^a + \omega^a{}_b \wedge e^b = \tfrac{1}{2}\,F_{bc}^{\cdot\cdot a}\,e^b \wedge e^c\,, \tag{4}$$

$$\Omega^a{}_b = d\,\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \tfrac{1}{2}\,R_{cdb}^{\cdot\cdot\cdot a}\,e^c \wedge e^d\,. \tag{5}$$

Definitions (4) and (5) agree with those of Trautman [4]. In (2) and the right hand side of (3) the Minkowski metric has been employed in the transvection of indices.

Whereas $L_{\text{rot}}$ is a perfect analog of the electromagnetic Lagrangian, $L_{\text{transl}}$ is not the simplest gauge-invariant Lagrangian one can think of. However, this fact may be regarded as just another peculiar feature of the gauge theory of the translation group. Indeed it is well known ([5], [6]) that the "minimal coupling" prescription for matter fields that is required for the invariance of the action under local translations is quite different from what is encountered in gauging internal symmetries and also the Lorentz group.

In [1] several physical reasons were advanced for the choice (2) of $L_{\text{transl}}$. In this note we point out that (2) can also be obtained by a purely formal guiding principle. To this end we list the simplest (as judged from the point of view of the calculus of differential forms) gauge-invariant translational Lagrangians:

$$L_0 = \tfrac{1}{4}\,e^a \wedge * e_a = e^0 \wedge e^1 \wedge e^2 \wedge e^3\,, \tag{6}$$

$$L_1 = \Theta^a \wedge * \Theta_a = \tfrac{1}{2}\,F_{bc}^{\cdot\cdot\,a}\,F^{bc}{}_{\cdot\cdot a}\,L_0\,, \tag{7}$$

$$L_2 = (\Theta^a \wedge e_a) \wedge * (\Theta^b \wedge e_b) = \tfrac{3}{2}\,F_{[abc]}\,F^{[abc]}\,L_0\,, \tag{8}$$

$$L_3 = (\Theta^a \wedge e^b) \wedge * (\Theta_a \wedge e_b) = 2\,L_1\,, \tag{9}$$

$$L_4 = (\Theta^a \wedge e^b) \wedge * (\Theta_b \wedge e_a) = \tfrac{1}{2}\,(F^{abc}\,F_{abc} - 2\,F_{ia}^{\cdot\cdot\,a}\,F^{ib}{}_{\cdot\cdot b}\,L_0\,. \tag{10}$$

Equation (10) coincides with (2) apart from a constant. We arrive at $L_4$ by requiring (i) that the Lagrangian 4-form be constructed from both $\Theta^a$ and $e^a$ *explicitly* (of course the tetrad is involved implicitly in the Hodge (*) operation), (ii) that it be "monomial" in these forms with only the operations "$\wedge$", "*" and contraction with the Minkowski metric being allowed, and (iii) that it be not the "square" of a scalar-valued form, if we define the "square" of a form $\omega$ by $\omega \wedge * \omega$. The last postulate excludes $L_2$, whereas $L_3$ is excluded by (i), since it is equivalent to $L_1$. Other Lagrangians obeying (i)–(iii) are more complicated than $L_4$ (e.g. they will involve $* \Theta^a$ instead of $\Theta^a$).

Physically, $L_0$ is peculiar in that it contains only the gauge potentials and thus yields only a "cosmological term" on the left hand side of the field equations. $L_1$ is the most straightforward analog of the electromagnetic Lagrangian, but appears to be unphysical, as the corresponding field equations do not have a Newtonian limit. Finally it is interesting to observe that the 3 invariants $L_2$, $L_3$, $L_4$ are basic and distinguished from the others by the property that they are directly related to the existence of physical solutions in the limit of teleparallelism ($\Omega^a{}_b = 0$ and hence $L_{\text{rot}} = 0$; com-

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pare the discussion in terms of the Weitzenböck invariants in [1]): Field equations admitting a Newtonian limit are obtained only from Lagrangians of the type

$$L_{\mathrm{N}} = \frac{1}{2l^2}(L_4 + c\,L_2)\,, \quad c \text{ arbitrary.} \tag{11}$$

The teleparallelism equivalent of Einstein's General Relativity is recovered by the choice $c = -1/2$. From the gauge point of view, however, $c = 0$ seems to be the most natural choice. Since (11) also admits the full Schwarzschild solution, the Lagrangian (1) does not contradict present day relativity experiments.

I thank Prof. Friedrich W. Hehl for helpful remarks.

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